Static and Dynamic properties of a Fermi-gas of cooled atoms near a wide Feshbach resonance.

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We introduce the Global Spin Model to study the static and dynamic properties of the ultracold fermionic gas near the broad Feshbach resonance. We show that the problem of molecular production, in a single-mode approximation, is reduced to the linear Landau-Zener problem for operators. The strong interaction leads to significant renormalization of the gap between adiabatic levels. In contrast to static problem the close vicinity of exact resonance does not play substantial role. Two main physical results of our theory is the high sensitivity of molecular production to the initial value of magnetic field and generation of a large BCS condensate distributed over a broad range of momenta in a wide range of parameters. We calculate the amplitude of the condensate as function of the initial detuning and the rate of the magnetic field sweep.

I. INTRODUCTION

In recent years there have been numerous achievements in the area of ultra-cold atomic physics. The major experimental tool for it is the use of the Feshbach resonances (FR)[1,2,3,4,5,6,7], which occurs when the energy of a quasibound molecular state becomes equal to the energy of two free alkali atoms. The magnetic-field dependence of the energy allows precise tuning of the atom-atom interaction strength in an ultracold gas[5]. Moreover, time-dependent magnetic fields can be used to reversibly convert atom pairs into weakly bound molecules[8,9,10,11,12,13,14,15]. This technique has proved to be extremely effective in converting degenerate atomic gases of fermions[9,13,14,15,16,17,18] and bosons[11,12,19] into bosonic dimer molecules.

The main theme of this paper is the crossover from the BCS to BEC condensates. The weakly and strongly paired fermionic s-wave superfluids were well understood by 60's, while the relation between the superfluids were studied later by Eagles[20], Leggett[21], and Noziéres and Schmitt-Rink[22]. These works treated the BCS state as a variational ground state and explicitly showed that there is no qualitative distinction between the BCS and BEC superfluids. After the discovery of High-T_c materials a possibility to describe the pseudogap phase as a long crossover between the BCS and BEC condensates renewed the interest to the phenomena[23,24]. It also become very important in the area of ultracold atomic gases where the transition/crossover can be directly observed in the molecular production experiments.

Theoretical works on the molecular production can be roughly referred to two categories. The first is a phenomenology suggesting that pairs of molecules independently undergo Landau-Zener (LZ) transitions[1,3]. The total number of molecules at the end of the process in these works is the LZ transition probability multiplied by the number of pairs. The most problematic issue in this approach is how to identify a pair in the gas of indistinguishable fermions. Direct calculation of the transition probability from a microscopic Hamiltonian,

to the 4-th order in the interaction constant[25], shows that, in contrast to the assumption of phenomenological works, the many-body effects are essential. Another category includes works based on a simplified model[26] in which molecules have one available state mimicking the condensate[27,28,29]. Although numerical works in this category display a reasonable temperature dependence, they give no clear physical picture and detailed parameter dependence.

In this article we consider a homogeneous gas in a box with the large volume V. The gas consists of Fermi isotopes of alkali atoms such as ^{6}Li, ^{24}Na, ^{40}K. The gas is subject to an external magnetic field such that the electronic Zeeman energy strongly exceeds the temperature, whereas the nuclear Zeeman energy remains much smaller than the temperature. Therefore, the electronic spins of atoms are almost completely polarized along the magnetic field. The nuclear spins in the ground state are then aligned with respect to the electronic spin by the hyperfine interaction. In a typical experiment[14,16] the magnetic field passes the Feshbach resonance.

The origin of this resonance is the following. When two atoms with one electron in the outer shell each turn into a molecule, the two electrons form a spin singlet. This requires one of the electrons to flip its spin, which costs the Zeeman energy for the electron. The resonance happens when Zeeman energy equals to the binding energy of the molecule. Neither exchange nor magnetic dipolar interaction of electrons can produce such a change of the spin state. This process is mediated by the hyperfine interaction. The characteristic scale of the hyperfine interaction ε_{hf} is in the range of mK and it is smaller or of the same order of magnitude as the molecular binding energy. It strongly exceeds the Fermi-energy of the gas which is in the range of $10^{-6}K$. Collisions in the s–wave channel of Fermi-atoms is allowed by the Pauli principle only if they have different states of nuclear spins. Therefore, experimenters create an admixture of about equal numbers of the fermionic atoms with different nuclear states, for example atoms of ^{40}K with the same total atomic spin 9/2 but different spin projection quantum numbers $-7/2$ and $-9/2$. Further we will describe this different states

by a pseudospin index σ accepting two values $\uparrow, \downarrow$.

The difference between the Zeeman energy and its resonance value is called detuning ϵ. It is the experimentally controlled parameter, which can be changed by changing external magnetic field. There are two different types of experimental procedures. In the first one, which we call static, an equilibrium or metastable state is established at a fixed value of ϵ and the number of atoms and molecules as well as the distributions of their velocities as function of detuning are studied. In the second one, which we call dynamic, the magnetic field sweeps passing the FR at which the detuning energy ϵ is zero. After passing through the resonance a part of atoms are transformed into diatomic bosonic molecules. The experimental and theoretical problem is the description of this process and finding the molecular production as a function of the magnetic field rate.

A major difference between these experiments is in time scales. At the first, static, type of experiment the system is always at thermal equilibrium (quasiequilibrium), while at the second, dynamic, type the gas is out of equilibrium – the equilibrating processes are too slow.

A general feature of the equilibrium and dynamic state is the appearance of the condensates: the Bardeen-Cooper-Schrieffer (BCS) condensate on atomic side of the resonance and the Bose-Einstein (BE) condensate on its molecular side. The condensates have the same symmetry. Therefore, the static transformation from the BCS to BE condensate proceeds not as a sharp transition, but rather as an eventual crossover.

The main subject of our paper is the BCS and BEC condensates densities and their correlations in both static and dynamic problems at condition of a broad resonance which will be shown to be equivalent to a strong interaction. As always the strong interaction problem is difficult to solve exactly. We propose a reasonable approximation which allows the analytical solutions of both, static and dynamic problems.

Below we introduce the necessary notations and definitions. The dependence of the s-scattering length a on external magnetic field B is determined by the well-known equation:

$$a(B) = a_0 \left(1 - \frac{B_1}{B - B_0}\right), \qquad (1)$$

where a_0 is the scattering length far from the resonance; B_0 is the resonance value of the field and B_1 is the magnetic-field width of the resonance which appear in experimental measurements. An intrinsic energy scale generated by the field width is[30]:

$$\tilde{\Delta} = \frac{4\mu_B^2 B_1^2 m a_0^2}{\hbar^2} \qquad (2)$$

We will show later that the value of $\tilde{\Delta}$ characterises the interaction in the Fermi-gas or the interaction of the BCS pairs with the BEC condensate.

There are two different regimes for both static and dynamic transitions: narrow and broad Feshbach resonance. The Feshbach resonance is narrow when $\tilde{\Delta}$ is much smaller than the Fermi energy ε_F and broad in the opposite case. Thus, the resonance is broad if the dimensionless parameter

$$\Gamma = \sqrt{\frac{\tilde{\Delta}}{\varepsilon_F}} \qquad (3)$$

is large. In the opposite case the resonance is narrow. The static and dynamics of the narrow resonance was thoroughly analyzed theoretically by Gurarie and Radzihovsky to whose review[30] we refer the reader. The case of the broad resonance was not analyzed since it is a strong interaction problem. We show that the condition of the broad resonance (10) allows simplification of the model despite of strong interaction between fermions. The key idea is a proper cut-off in the momentum space and neglect of the fermion dispersion. The resulting model is similar to the Dicke spin model for superradiance[31]. This model allows us to solve the static problem exactly. The complete spectrum and eigenstates are found. The solution displays a crossover from BCS to BEC in a range of detuning near the FR. We find the density of the condensates and their correlation as function of the detuning. We show that, contrary to the standard BCS solution, the scale of momenta in which the Cooper pair wave function is confined is $(2m\tilde{\Delta})^{1/2}$, much larger than the Fermi energy. Therefore, the size of the Cooper pair in real space is much less than the average distance between atoms. We discuss consequences of this fact for both statics and dynamics.

The dynamic problem is reduced to the Landau-Zener problem in the operator form which also can be solved exactly. One of conservation laws in our theory ensures that the increase of one of condensates proceeds only at expense of another. The main conclusions of the dynamic problem are: i) Sensitivity of the molecular production to the initial value of the BCS condensate; ii) Conservation of a sum of weighted BCS and BEC condensate densities which indicates that one of condensates can grow only at the expense of another one. The dynamic problem was briefly described earlier in our article[32].

II. THE MODEL AND CONSERVATION LAWS

We start with the Timmermans[7] Hamiltonian[42]:

$$\hat{H} = \sum_{\mathbf{p},\sigma}(\epsilon + \varepsilon_{\mathbf{p}})\hat{a}_{\mathbf{p}\sigma}^\dagger \hat{a}_{\mathbf{p}\sigma} + \sum_p \omega_{\mathbf{q}}\hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}} +$$
$$\frac{g}{\sqrt{V}} \sum_{\mathbf{p},\mathbf{q}} \left(\hat{b}_{\mathbf{q}}\hat{a}_{\mathbf{p}+\mathbf{q}\uparrow}^\dagger \hat{a}_{-\mathbf{p}\downarrow}^\dagger + \hat{b}_{\mathbf{q}}^\dagger \hat{a}_{-\mathbf{p}\downarrow}\hat{a}_{\mathbf{p}+\mathbf{q}\uparrow}\right) \qquad (4)$$

Here $\hat{a}_{p\sigma}^\dagger$ are creation operators of fermionic atoms with momentum $\mathbf{p}$, pseudospin σ, and kinetic energy $\varepsilon_{\mathbf{p}} =$

$\mathbf{p}^2/2m$; $\hat{b}_{\mathbf{q}}^{\dagger}$ are the creation operators of the bosonic diatomic molecule with kinetic energy $\omega_{\mathbf{q}} = \mathbf{q}^2/4m$; the position of the FR is controlled by the experimentally tunable detuning energy ϵ, wich becomes time-dependent in the dynamic problem. The last term in (4) is the interaction with a coupling constant g. It describes the formation of a molecule from two atoms and the inverse process of dissociation into two atoms.

As it was explained earlier, the coupling constant g is provided by the hyperfine interaction. Its value can be estimated[33] as $g \sim \varepsilon_{hf}\sqrt{a_m^3}$, where ε_{hf} is a characteristic hyperfine energy (about 1mK) and a_m is the size of the diatomic molecule. The Hamiltonian (4) neglects nonresonant direct atom-atom and molecule-molecule interactions that near any FR are subdominant to the resonant scattering. To some extent the direct atom-atom interaction is taken into account by the value a_0 in eq. (1). We use the word "atoms" for both uncoupled atoms and atoms within molecules, and the word "fermions" for atoms that are not bound in molecules. Correspondingly, we denote the number of atoms as N and the number of fermions as $\hat{N}_F = \sum_{\mathbf{p},\sigma} \hat{a}_{\mathbf{p},\sigma}^{\dagger} \hat{a}_{\mathbf{p},\sigma}$. Since the diatomic molecule can be created by the interaction term in (4) only at expense of two fermions the Hamiltonian (4) conserves the total number of atoms:

$$N = \hat{N}_F + 2\sum_{\mathbf{q}} \hat{b}_{\mathbf{q}}^{\dagger} \hat{b}_{\mathbf{q}} \qquad (5)$$

Therefore, in the Hamiltonian (4) the detuning energy can be transferred with the coefficient -2 to the energy of molecules. To find the connection between the coupling constant g in the Hamiltonian (4) and the value $\tilde{\Delta}$ defined by equation (2) we use the perturbation theory to the second order to eliminate $\hat{b}_{\mathbf{q}}$ and $\hat{b}_{\mathbf{q}}^{\dagger}$ from the static Hamiltonian (4) and obtain the Hamiltonian for fermions only with 4-fermion interaction. Assuming that $\varepsilon_{\mathbf{p}}$ and $\omega_{\mathbf{q}}$ can be neglected in comparison to ϵ (this assumption will be justified later), the interaction Hamiltonian reads:

$$H_{int} = \frac{g^2}{2V\epsilon} \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} \hat{a}_{\mathbf{p}+\mathbf{q}\uparrow}^{\dagger} \hat{a}_{-\mathbf{p}\downarrow}^{\dagger} \hat{a}_{-\mathbf{p}'\downarrow} \hat{a}_{\mathbf{p}'+\mathbf{q}\uparrow} \qquad (6)$$

Thus, the effective interaction constant for fermions is $g_F = g^2/2\epsilon$. It is negative (attraction) at negative ϵ. The s-scattering length a is related to the interaction constant g_F by a standard relation[34] $g_F = 4\pi\hbar^2 a/m$. Thus, the singular part of the scattering length is associated with the fermion-boson coupling constant g and the detuning energy ϵ as follows:

$$a(\epsilon) = \frac{mg^2}{8\pi\hbar^2\epsilon} \qquad (7)$$

Comparing equation (7) with the resonance term in equation (1) and identifying $\epsilon = \mu_B (B - B_0)$, we arrive at a relation $2B_1 a_0 \mu_B = mg^2/4\pi\hbar^2$. Using $\epsilon_F = \hbar^2 (3\pi^2 n)^{2/3}/2m$, where n is the density of fermions, and

substituting this into equations (2,3) we express the intrinsic energy $\tilde{\Delta}$ and dimensionless parameter Γ in terms of the coupling constant g:

$$\tilde{\Delta} = \frac{m^3 g^4}{16\pi^2\hbar^6} \qquad (8)$$

$$\Gamma = \frac{m^2 g^2}{\hbar^4 n^{1/3}} \frac{1}{\pi^{5/3} 3^{1/3} 2^{3/2}} \qquad (9)$$

The Hamiltonian (4) is still too complicated. We show that the broad resonance condition

$$\Gamma \gg 1 \qquad (10)$$

allows us to apply two simplifying assumptions.

The first one of them is the single-mode approximation: we neglect all bosonic modes except the one with zero momentum. The characteristic energy scale in statics and dynamics is $\tilde{\Delta}$. The corresponding value of the scattering length following from equation (8) at such a detuning is $a(\tilde{\Delta}) \sim n^{-1/3}/\Gamma$, which gives the characteristic value of the gas parameter $an^{1/3} \sim \Gamma^{-1} \ll 1$. Small $|\epsilon| \sim \tilde{\Delta}/\Gamma$ correspond to the strong interaction regime in which the fermions are distributed in a broad range of momentum $\sim \sqrt{2m\tilde{\Delta}}$. The maximal modulus of the scattering amplitude f for such particles achieved in the unitary limit is $f \sim \hbar/\sqrt{2m\tilde{\Delta}}$. The corresponding value of the gas parameter $fn^{1/3}$ is again of the order $\Gamma^{-1} \ll 1$. At this conditions and at zero temperature almost all particles of the Bose-gas fall into the coherent condensate with zero momentum. We thus will substitute $\hat{b}_{\mathbf{q}} = \delta_{0,\mathbf{q}} b_0$ in the Hamiltonian (4). This is the single-mode approximation.

The applicability of the single-mode approximation to the dynamics seems less obvious since the formation of the condensate in the Bose gas requires some time which must be compared with the characteristic time of the fermion-boson transition. The kinetic of the condensate formation was theoretically studied by Kagan et al.[35,36] and by Gardiner et al.[37]. In the article[35] it was concluded that the formation of condensate starts after some cooling process which requires a delay time, typically few tens of ms in the laser cooled gases and then the condensate grows rapidly. In the experiment[38] even smaller delay time was found. The characteristic time of sweeping through the FR was of the same order of magnitude or longer. Besides, the experimenters started the sweeping in a close vicinity of the resonance in which at least the BCS condensate was sufficiently strong. As we will see later, in dynamics it produces the BEC condensate during the time characteristic for the Landau-Zener processes.

The second approximation we apply is disregarding the fermionic dispersion in (4) in comparison to ε. A typical value of b_0 is $\sqrt{N}$. In the broad resonance approximation $g\sqrt{N}/\sqrt{V} = g\sqrt{n}$ is much larger than $\varepsilon_F \sim \hbar^2 n^{2/3}/m$, $gm\sqrt{n}/\hbar^2 n^{2/3} \sim \sqrt{\Gamma} \gg 1$, see (9). With this motivation we neglect the kinetic energy of fermions in comparison to ε. Though rather rough, this approximation retain essential features of the exact Hamiltonian. The character

and precision of this approximation will be discussed in more details later.

These two approximations greatly simplify the Hamiltonian (4) reducing it to the following form:

$$H_{sm} = \sum_{\mathbf{p},\sigma}^{p_s} \epsilon \hat{a}_{\mathbf{p}\sigma}^\dagger \hat{a}_{\mathbf{p}\sigma} + \tag{11}$$

$$\frac{g}{\sqrt{V}} \left[b_0 \sum_{\mathbf{p}}^{p_s} a_{\mathbf{p}\uparrow}^\dagger a_{-\mathbf{p}\downarrow}^\dagger + b_0^\dagger \sum_{\mathbf{p}}^{p_s} a_{\mathbf{p}\downarrow} a_{-\mathbf{p}\uparrow} \right]$$

The number of atoms conserved by the Hamiltonian (11) reads:

$$N = \sum_{\mathbf{p},\sigma}^{p_s} \hat{a}_{\mathbf{p}\sigma}^\dagger \hat{a}_{\mathbf{p}\sigma} + 2b_0^\dagger b_0 \tag{12}$$

In Eqs. (11) and (12) we have introduced the cutoff momentum p_s to account for the dropped fermionic dispersion. The value of this momentum must be such that our approximation of neglecting the kinetic energy in comparison to $\tilde{\Delta}$ is justified. Thus, a reasonable cut-off momentum is:

$$p_s = \sqrt{2m\tilde{\Delta}} \tag{13}$$

Plugging eq. (8) into eq. (13), we find:

$$p_s = \frac{m^2 g^2}{2^{3/2}\pi\hbar^3} \tag{14}$$

The number of available states N_s, which includes all the states with $p < p_s$, reads:

$$N_s = V \frac{p_s^3}{3\pi^2\hbar^3} \tag{15}$$

Eq. (14) gives the following result for the density of the available states $n_s = N_s/V$:

$$n_s = \frac{m^6 g^6}{2^{9/2} \cdot 3\pi^5\hbar^{12}} \tag{16}$$

With precision of the coefficient 1.09 the value

$$\Delta = g\sqrt{n_s} \tag{17}$$

coincides with $\tilde{\Delta}$. The condition of the broad resonance is equivalent to the strong inequality:

$$N_s \gg N \tag{18}$$

Let us consider the whole Hilbert space as a set of pairs of conjugated states $|\mathbf{p},\uparrow\rangle$ and $|-\mathbf{p},\downarrow\rangle$. Each pair can be either empty, or single occupied, or doubly occupied. A single occupied pair is not changed by the Hamiltonian (11). The whole Hilbert space can then be split on two invariant subspaces: the set of the single occupied pairs and the rest, where no state is single occupied. The Hamiltonian is diagonal in the first subspace, so it induces no transitions in it. We thus consider only an invariant subspace of the Hilbert space with all pairs either empty or doubly occupied.

Besides of the occupation number of fermionic pairs, the state is fixed with the number of bosonic molecules N_m. The Hilbert subspace invariant under the action of the Hamiltonian (11) at a fixed total value of atoms N consists of states with given number N_m of molecules in the condensate and the rest of atoms occupying pairs of fermionic states. The number of occupied pairs is $M = N/2 - N_m$. Summing the number of states with different possible M, we find for the dimensionality of the Hilbert space $\mathcal{D}(N, N_s)$ with given N and N_s:

$$\mathcal{D}(N, N_s) = \sum_{M=0}^{N/2} \frac{N_s!}{M!(N_s - M)!} \tag{19}$$

III. THE GLOBAL SPIN MODEL AND ITS HILBERT SPACE

Following Anderson[39], we introduce spin operators:

$$s_{\mathbf{p}z} = \frac{1}{2} \left(a_{\mathbf{p}\uparrow}^\dagger a_{\mathbf{p}\uparrow} + a_{-\mathbf{p}\downarrow}^\dagger a_{-\mathbf{p}\downarrow} - 1 \right) \tag{20}$$

$$s_{\mathbf{p}+} = a_{-\mathbf{p}\downarrow}^\dagger a_{\mathbf{p}\uparrow}^\dagger; \qquad s_{\mathbf{p}-} = a_{\mathbf{p}\uparrow} a_{-\mathbf{p}\downarrow} \tag{21}$$

In the double-occupied or empty fermionic pairs subspace they obey the standard spin-1/2 commutation relations:

$$[s_{\mathbf{p}z}, s_{\mathbf{p}'\pm}] = \pm\delta_{\mathbf{p}\mathbf{p}'}s_{\mathbf{p}\pm}; \quad [s_{\mathbf{p}+}, s_{\mathbf{p}'-}] = 2\delta_{\mathbf{p}\mathbf{p}'}s_{\mathbf{p}z} \tag{22}$$

so that the double occupied pair corresponds to $s_{\mathbf{p}z} = +\frac{1}{2}$ and empty pair corresponds to $s_{\mathbf{p}z} = -\frac{1}{2}$. Note that a single-occupied fermionic pair corresponds to a singlet spin state: all three spin operators (20, 21) turn such a pair to zero.

The neglect of kinetic energy allows us to rewrite the Hamiltonian (11) in terms of only global spin operators:

$$H_S = 2\epsilon S_z + \frac{g}{\sqrt{V}} \left(b_0 S_+ + b_0^\dagger S_- \right) \tag{23}$$

(we have omitted a constant originated from the term -1 in equation (20)), where the global spin components

$$S_z = \sum s_{\mathbf{p}z}; \qquad S_\pm = \sum s_{\mathbf{p}\pm} \tag{24}$$

obey the standard commutation relationships: $[S_z, S_\pm] = \pm S_\pm$; $[S_+, S_-] = 2S_z$. We will call the model with the Hamiltonian (23) the Global Spin Model (GSM).

A subtle assumption incorporated in the derivation of the Hamiltonians (11), (23), and in the permutation relations for the global spin components is that the summations in the two definitions (24) run over the same range of momenta. It is not obvious since the summation in the sum for S_z is limited by the condition $\varepsilon_{\mathbf{p}} < |\epsilon|$, whereas the summation in the second sum originated

from the boson-fermion interaction part of the Hamiltonian is naturally cut off by the range of interaction. It means that g is a function of momentum vanishing at sufficiently large values of the momentum modulus. In the dynamical problem the Anderson spins rotate with the frequencies $(\epsilon + \varepsilon_{\mathbf{p}})/\hbar$. Therefore they rotate coherently and enhance remarkably their effective field exerted to the moleculare amplitude b_0 only if $\varepsilon_{\mathbf{p}} < |\epsilon|$. We will see that such a coherence indeed takes place in the dynamic problem. The contributions from larger values of momenta to $S_\pm$ are incoherent and mutually compensate their effect. Thus, for dynamic problem, the summation in the same momentum region in the two equations (24) is justified.

It is not so obvious for the static problem. We will show that the static GSM has many qualitative features resembling what is expected for the initial Timmermanns[7] *et al.* model and observed in the experiment, though nobody solved the latter model exactly. In particular the GSM displays a crossover from BCS to BEC condensate with a large gap due to strong interaction in a broad vicinity of the Feshbach resonance. However, in quantitative details they are different. Besides, the GSM (23) has an additional symmetry and an additional conserved value which the Timmermans Hamiltonian does not possess. Indeed the Hamiltonian (23) conserves not only the value of

$$ Q = S_z + b_0^\dagger b_0, \qquad (25) $$

equivalent to the number of atoms (12), but also the total spin S, where:

$$ S(S+1) = S_z^2 + \frac{1}{2}(S_+ S_- + S_- S_+) \qquad (26) $$

In the thermodynamic limit of large system it is possible to neglect 1 in comparison to S and consider S_+ and S_- as commutative values. The conservation laws (26, 25) imply that the quantity of one of the condensates (BCS or BEC) can be increased only at the expense of the other.

The cut-off in the momentum space define a finite-dimensional Hilbert space of states R_N. What we call the GSM is this Hilbert space and the Hamiltonian (23) acting in it.

Each state in R_N is a vector from a direct product of N_s spin 1/2 representations corresponding to the Anderson spins $\mathbf{s_p}$. The combinatorial coefficient $N_s!/M!(N_s - M)!$ entering eq. (19) can be treated as the number of possible distributions of M spins up and $N_s - M$ spins down, i.e. the number of different states with the spin projection:

$$ S_z = M - \frac{N_s}{2} \qquad (27) $$

Since $0 \le M \le N/2$, the projection S_z at a fixed S is limited by inequalities:

$$ -S \le S_z \le (N - N_s)/2. \qquad (28) $$

According to (18), S_z is always large and negative. The total spin S cannot be smaller than $|S_z|$, it also cannot be larger than $N_s/2$. Thus, the allowed values of the total spin S are $(N_s - N)/2 \le S \le N_s/2$.

The number $\mathcal{N}(S, N_s)$ of different multiplets with a fixed value of the total spin S which appear at the addition of N_s spins 1/2 reads[33]

$$ \mathcal{N}(S, N_s) = \frac{N_s!\,(2S+1)}{\left(\frac{N_s}{2} - S + 1\right)!\left(\frac{N_s}{2} + S\right)!} \qquad (29) $$

Each of these representations contains generally $2S + 1$ states, but only $S - \frac{N_s - N}{2}$ of them are allowed by the inequalities (28). The number of states $\tilde{\mathcal{N}}(S, S_z)$ in the Hilbert space of the GSM with fixed values S and S_z reads:

$$ \tilde{\mathcal{N}}(S, S_z) = \theta(S_z + S)\theta\left(-\frac{N_s}{2} + \frac{N}{2} - S_z\right)\mathcal{N}(S), \quad (30) $$

where $\theta(x)$ is the Heaviside step function.

The Hamiltonian (23) formally coincides with the famous Dicke Hamiltonian for the so-called superradiance problem[31], but it acts in a different Hilbert space.

IV. SPECTRA AND EIGENSTATES OF THE GSM

The Hamiltonian (23) commutes with S and N. Therefore, its stationary states $|\Psi\rangle$ at fixed values N_s, N, and S can be represented by a superposition of states with fixed number of molecules N_m or $S_z = \frac{N - N_s}{2} - N_m$:

$$ |\Psi\rangle = \sum_{N_m = 0}^{\frac{N - N_s}{2} + S} \Psi_{N_m} |N_m\rangle, \qquad (31) $$

whose amplitudes Ψ_{N_m} obey the stationary Schrödinger equation:

$$ \begin{aligned} E\Psi_{N_m} = &-2\epsilon N_m \Psi_{N_m} + \\ &\frac{g}{\sqrt{V}}\sqrt{N_m(S - S_z)(S + S_z + 1)}\,\Psi_{N_m - 1} + \quad (32) \\ &\frac{g}{\sqrt{V}}\sqrt{(N_m + 1)(S + S_z)(S - S_z + 1)}\,\Psi_{N_m + 1} \end{aligned} $$

This system is still rather complicated, but it is strongly simplified by the broad resonance condition (18). Indeed, due to this inequality $S \approx -S_z \approx N_s/2$, and it is possible to replace $S - S_z$ and $S - S_z + 1$ in eq. (33) by N_s. We should be more careful with the expression $S + S_z$ since the two terms almost completely cancel each other. Employing the notation defined earlier $\Delta = g\sqrt{n_s}$ (17) and introducing new variables

$$ S_z = s - S - m; \quad s = \frac{S}{2} + \frac{N - N_s}{4}; \quad m = N_m - s, \quad (33) $$

we arrive at a simplified version of equations (33):

$$-2\epsilon m\Psi_m + \Delta\sqrt{(s-m)(s+m+1)}\Psi_{m+1} +$$
$$\Delta\sqrt{(s+m)(s-m+1)}\Psi_{m+1} = (E+2s\epsilon)\Psi_m, (34)$$

Since $N_m \geq 0$ and $S + S_z \geq 0$, the quantum number m runs from $-s$ to s and $0 \leq s \leq N/4$. The equation (34) is easily recognizable as generated by a reduced spin Hamiltonian:

$$H_r = -2\epsilon s_z + 2\Delta s_x \qquad (35)$$

where s_x, s_z are spin operators corresponding to the reduced total spin s. This is a Hamiltonian of a spin s in the magnetic field $2\sqrt{\epsilon^2 + \Delta^2}$ tilted in the xz plane at the angle $\theta = -\tan^{-1}(\Delta/\epsilon)$ to the z axis.

The energy levels are labeled by two integers s and $\tilde{m}$ (we have introduced tilde to distinguish projection to the axis tilted by the angle θ to z−axis from projection to the z−axis):

$$E_{s\tilde{m}}(\epsilon) = 2\tilde{m}\sqrt{\epsilon^2 + \Delta^2} - 2s\epsilon \qquad (36)$$

The spectrum (36) possesses a symmetry:

$$E_{s\tilde{m}}(\epsilon) = -E_{s,-\tilde{m}}(-\epsilon) \qquad (37)$$

Levels with the same s and different $\tilde{m}$ do not cross, but the levels with different s and $\tilde{m}$ cross each other. The crossing of the levels $(s, \tilde{m})$ and $(s', \tilde{m}')$ happens at a point:

$$\epsilon = \Delta \frac{\text{sign}\left(\frac{s-s'}{\tilde{m}-\tilde{m}'}\right)}{\sqrt{\left(\frac{s-s'}{\tilde{m}-\tilde{m}'}\right)^2 - 1}}, \quad |s-s'| > |\tilde{m} - \tilde{m}'| \qquad (38)$$

Besides of crossings each level $(s, \tilde{m})$ at any ϵ is $\mathcal{N}(S, S_z)$ fold degenerate as it is determined by eq. (29). In terms of reduced spin variables it reads:

$$\mathcal{N}(N, s, \tilde{m}) \approx \theta(s - \tilde{m})\theta(s + \tilde{m})\frac{N_s^{N/2 - 2s + 1}e^{-N/2 + 2s}}{\sqrt{2\pi}(N/2 - 2s)!} \qquad (39)$$

For a fixed s the state with minimal energy is $(s, -s)$. The ground state corresponds to the maximal possible value $s = N/4$ and $\tilde{m} = -s$. Its energy is:

$$E_G = E_{N/4, N/4} = -\frac{N}{2}\left(\sqrt{\epsilon^2 + \Delta^2} + \epsilon\right) \qquad (40)$$

The energy of the ground state is separated from the rest of spectrum by a finite energy gap δ, which is determined by the following equation:

$$\delta = 2\left[\sqrt{\epsilon^2 + \Delta^2} + \epsilon\theta(-\epsilon)\right]. \qquad (41)$$

The first excited state at negative ϵ is the state with $s = \frac{N}{4} - 1$ and $\tilde{m} = -\frac{N}{4} + 1$; at positive ϵ it is the state with $s = N/4$ and $\tilde{m} = -\frac{N}{4} + 1$. In the

ground state the maximal spin $s = N/4$ is oriented in the xz−plane at the angle $\theta = \tan^{-1}(\Delta/\epsilon)$ to the z−axis. This state can be thought of as a set of $N/2$ spins $1/2$, all directed along the same line. The corresponding wave function is $|\psi\rangle = \prod_{i=1}^{N/2} \otimes |\psi_i\rangle$, where $|\psi_i\rangle = (\cos\theta/2, \sin\theta/2)_i^T$. The operator s_z is represented by the direct sum: $s_z = \sum_{i=1}^{N/2} s_z^i$. Therefore, the average value of m in the ground state is equal to $\langle m\rangle_G = \langle\psi|s_z|\psi\rangle = \sum_{i=1}^{N/2}\langle\psi_i|s_z^i|\psi_i\rangle = \frac{N}{4}\cos\theta = N\epsilon/4\sqrt{\epsilon^2 + \Delta^2}$. Employing the third equation (33), we find the average number of molecules in the ground state:

$$\langle N_m\rangle_G = \frac{N}{4}\left(1 + \frac{\epsilon}{\sqrt{\epsilon^2 + \Delta^2}}\right) \qquad (42)$$

It smoothly varies from 0 at $\epsilon = -\infty$ to $N/2$ at $\epsilon = +\infty$. The width of the transition is determined by Δ. The value $\langle N_m\rangle_G - \frac{N}{4}$ is an odd function of the detuning energy. The average number of fermions $\langle N_F\rangle$ can be found from the conservation law $N_F + 2N_m = N$.

The fluctuation of the number of molecules in the ground state also can be calculated exactly. Indeed, $\langle(\Delta N_m)^2\rangle_G = \langle\psi|(s_z)^2|\psi\rangle - \langle\psi|s_z|\psi\rangle^2$ using $\langle\psi|(s_z)^2|\psi\rangle = N/8 + \sum_{i\neq j}^{N/2}\langle\psi_i|s_z^i|\psi_i\rangle\langle\psi_j|s_z^j|\psi_j\rangle = N/8 + \cos^2(\theta)N(N-2)/16$. Thus, at large N, the quadratic fluctuation of the number of molecules reads:

$$\langle(\Delta N_m)^2\rangle_G = \frac{N}{8}\frac{\Delta^2}{\epsilon^2 + \Delta^2} \qquad (43)$$

The fluctuation $\langle(\Delta N_m)^2\rangle$ is maximal at $\epsilon = 0$ — at the FR, and is an even function of the detuning ϵ.

An important value is the BCS condensate amplitude $\langle S_+\rangle_G$ in the ground state. Using the first equation (33) and (26), we find that $|\langle S_+\rangle_G|^2 \approx S^2 - \langle S_z^2\rangle_G \approx (N_s/2)^2 - (N_s/2 - N/4)^2 - N_s\langle s_z\rangle_G \approx \frac{1}{4}NN_s(1 - \cos\theta)$ and thus

$$|\langle S_+\rangle_G| = \sqrt{\frac{NN_s}{2}}|\sin\theta/2| \qquad (44)$$

The ground state is symmetric with respect to any permutation of the N_s pair states, which is the Hamiltonian symmetry operation. Therefore, all amplitudes $\langle a_{\mathbf{p}\uparrow}^\dagger a_{-\mathbf{p}\downarrow}^\dagger\rangle$ are equal and each of them is equal to $\sqrt{N/N_s}\sin(\theta/2)$. A simple physical picture behind this distribution is the following. The total number of occupied pairs is equal to $N\sin^2(\theta/2)$. They are equally distributed among N_s available states. Therefore, the probability to find a pair at any fixed state is $N\sin^2(\theta/2)/N_s$ and amplitude to find this state is $\sqrt{N/N_s}\sin(\theta/2)\times e^{i\varphi_\mathbf{p}}$. This result coincides with indicated above if all amplitudes are coherent and have the same phase. According to this picture only the total number of pairs depend on detuning, but their distribution in momentum space remains unchanged: they are uniformly distributed over momenta from zero to p_s. It means that the size of the Cooper pair is always $\hbar/p_s$. it is much smaller than the

distance between gas particles $n^{-1/3}$. The pairs are compact, but still the collective interaction of the fermions inside the pairs play important role. The implicit assumption of the theory is that the size of the Bosonic molecule is much less than $\hbar/p_s$. Another important difference between the Cooper pair and molecule is that in the former electron spins of atoms are parallel being polarized by magnetic field, whereas in the latter the electron spins are antiparallel. The broad distribution of the Cooper wave function in the momentum space is a consequence of the corresponding broad range of the hyperfine interaction. This property will persist in a more accurate theory which does not accept our simplifying approximations.

We notice from (42) and (44) that the value

$$N_s N_m + |\langle S_+ \rangle|^2 = N_s N/2 \qquad (45)$$

is a detuning independent constant.

Finally, we find the BEC-BCS correlation function $\langle b_0 S_+ \rangle$. We demonstrated earlier (see the derivation of eq. (34) from (33)) that in the broad resonance approximation $N_s \gg N$ the product of operators $b_0 S_+$ can be replaced by $\sqrt{N_s} s_+$. Thus,

$$\langle b_0 S_+ \rangle = \frac{\sqrt{N_s} N}{4} \sin \theta = \frac{\sqrt{N_s} N}{4} \frac{\Delta}{\sqrt{\epsilon^2 + \Delta^2}} \qquad (46)$$

This correlator vanishes at $\epsilon = \pm\infty$ and has maximum at $\epsilon = 0$. An interesting relationship between the BEC-BCS correlator and the fluctuation of the number of molecules follows from eqns. (43) and (46): $N_s N \langle (\Delta N_m)^2 \rangle_G = 2 \langle b_0 S_+ \rangle^2$.

The GSM displays BCS-BEC crossover in the range $|\epsilon| \sim \Delta$ near the Feshbach resonance. In this range the BCS condensate amplitude grows to the value $\sim \sqrt{N_s N/2} \gg N/2$. This enhancement of the condensate is due to the distribution of the Cooper pairs over a wide range of momenta strongly exceeding the Fermi sphere. It indicates that the famous BCS exponentially small condensate does not appear even at very large detuning exceeding Δ. In the BCS theory the condensate or energy gap is exponentially small not only due to the weakness of interaction, but also because the attraction range in the momentum space is very narrow. The latter condition is violated not only in the GSM, but also in the initial Timmermanns *et al.* model.

V. DYNAMIC ATOMS-TO MOLECULES TRANSFORMATION

In this section we consider the transformation of Fermi-atoms into molecules in the cold Fermi gas under the sweep of the magnetic field. In the Hamiltonian (11) the parameter ϵ is driven by magnetic field: $\epsilon = \mu_B(H - H_0)$. At magnetic field sweeping, the detuning energy ϵ depends on time. The time-dependent Hamiltonian regulating the process of atoms-to-molecules transformation reads:

$$\hat{H} = 2\epsilon(t)\hat{S}_z + \frac{g}{\sqrt{V}} \left(\hat{b}\hat{S}_+ + \hat{b}^\dagger \hat{S}_- \right) \qquad (47)$$

It formally coincides with the GSM Hamiltonian (23) with the only difference that the detuning energy ϵ depends on time. Even the time-dependent Hamiltonian (47) commutes both with the operator Q given by (25), equivalent to the operator of the total number of atoms N, and with the square of the total spin operator S^2 (26) which both remain the integrals of motion. The Heisenberg equations of motion are:

$$\hbar\dot{\hat{b}} = -i\tilde{g}\hat{S}_-; \qquad \hbar\dot{\hat{S}}_- = -2i\epsilon(t)\hat{S}_- + 2i\tilde{g}\hat{b}^\dagger\hat{S}_z \qquad (48)$$

where $\tilde{g} = g/\sqrt{V}$ and dots denote the time derivatives. Generally these equations are non-linear. However, in the broad-resonance approximation $S_z = -N_s/2$, they become linear. Eliminating S_-, we arrive at an ordinary linear differential equation for the operator $\hat{b}$:

$$\hbar^2\ddot{\hat{b}} + 2i\hbar\epsilon(t)\dot{\hat{b}} + \Delta^2\hat{b} = 0 \qquad (49)$$

where Δ is defined in (17).

Equation (49) becomes the parabolic cylinder equation if $\epsilon(t)$ is a linear function of time. In the LZ theory it describes the evolution of the probability amplitude to find the system in one of its two states. The value Δ in the corresponding LZ problem is the matrix element of transition between the two crossing terms, the so-called LZ gap.

As we already argued, during the magnetic field sweep the Anderson spins with $p \ll p_s$ rotate coherently with almost the same angular velocity and enhance the effective field acting on the BEC condensate. The spins with $p > p_s$ rotate with substantially different angular velocities and mutually cancel their contribution to the effective field. Our model coarsens these features, neglecting the decoherence of spins with $p < p_s$ and completely ignoring spins with $p > p_s$.

An important conclusion is that strong interaction renormalizes the LZ gap. The energy scale which appears in perturbation theory is $\Delta^{(0)} = g\sqrt{n}$[25]. For a broad resonance $\Delta = g\sqrt{n_s}$ is much larger than $\Delta^{(0)}$ and does not depend on the atomic density.

Employing equations (48), the general solution of the ordinary differential equation (49) reads (further we put $\hbar = 1$):

$$\hat{b}(t) = u(t,t_0)\hat{b}(t_0) - i\tilde{g}v(t,t_0)\hat{S}_-(t_0), \qquad (50)$$

$$i\tilde{g}\hat{S}_-(t) = -\dot{u}(t,t_0)\hat{b}(t_0) + i\tilde{g}\dot{v}(t,t_0)\hat{S}_-(t_0), \qquad (51)$$

where $u(t,t_0)$ and $v(t,t_0)$ are standard solutions of the same equation (49) satisfying the initial conditions $u(t_0,t_0) = 1$, $\dot{u}(t_0,t_0) = 0$ and $v(t_0,t_0) = 0$, $\dot{v}(t_0,t_0) = 1$. These solutions have the following properties:

$$|u|^2 + \Delta^{-2}|\dot{u}|^2 = \Delta^2|v|^2 + |\dot{v}|^2 = |u|^2 + \Delta^2|v|^2 = 1; \quad (52)$$

$$\dot{u}^*\dot{v} + \Delta^2 u^* v = 0; \quad \dot{u}v - u\dot{v} = e^{-i\int_{t_0}^t \epsilon(t')dt'}. \quad (53)$$

The solution (50,51) completely determines the evolution of the number of molecules $N_m(t) = \langle \hat{b}^\dagger \hat{b} \rangle(t)$, the BCS condensate amplitude $F(t)$ defined by equation $F^2(t) \equiv \langle \hat{S}_+ \hat{S}_- \rangle(t)$, and the BCS-BEC coherence factor $C(t) \equiv \langle \hat{b}^\dagger \hat{S}_- \rangle(t)$. It is convenient to introduce the intensive values: the molecular (or BEC) density $n_m(t) = N_m(t)/V$, the BCS condensate density $f(t) = g\sqrt{F(t)}/V$ and the BEC-BCS correlator density $c(t) = -igC(t)/V^{3/2}$. Then equations of motion read

$$n_m(t) = |u|^2 n_{m0} + |v|^2 f_0^2 + 2\mathrm{Re}\,(u^* v c_0) \quad (54)$$

$$f^2(t) = |\dot{u}|^2 n_{m0} + |\dot{v}|^2 f_0^2 + 2\mathrm{Re}\,(\dot{u}^* \dot{v} c_0) \quad (55)$$

$$c(t) = \dot{u}u^* n_{m0} + \dot{v}v^* f_0^2 + u^* \dot{v} c_0 + \dot{u}v^* c_0^*, \quad (56)$$

where we also introduced $n_{m0} = n_m(t_0)$, $f_0 = f(t_0)$, and $c_0 = c(t_0)$ – the initial values of the density of molecules, the BSC condensate density, and the BEC-BCS correlator density. Using (52) and summing eqs. (54) and (55), we find:

$$\Delta^2 n_m(t) + f^2(t) = \mathrm{const}, \quad (57)$$

which is a consequence of the conservation laws. Since for any state $F^2(t) > 0$, if there are no molecules in the initial state, their number $N_m(t)$ can not exceed the value $F^2(t_0)/N_s$ at any time. Although equation (57) has the same meaning as equation (45) it is more general as it shows that one of condensates can be extended only at the expense of another *even at a finite sweep rate*.

Below we consider two experimentally most relevant situations: i) only fermions and no molecules; ii) only molecules and no fermions in the initial state. In both these cases the initial value $C = \langle \hat{b}^\dagger \hat{S}_- \rangle(t_0) = 0$. In the case of no molecules in the initial state, so $N_m(t_0) = 0$, the general equations (54,56) simplify to

$$N_m(t) = \tilde{g}^2 |v|^2 F^2(t_0); \quad i\tilde{g}C(t) = -\tilde{g}^2 \dot{v}v^* F^2(t_0) \quad (58)$$

The evolution of $F(t)$ in this case is determined by (57) and (58). Note that the coherence factor $C(t)$ does not remain zero.

In order to produce a reasonable fraction of molecules it is necessary to have a large condensate amplitude in the initial state with the only exception for the equilibrium initial state and adiabatically slow sweep of magnetic field. Due to the finite gap in the spectrum our model predicts that in this situation the system will adiabatically follow the ground state. However, the initial state of the gas in the experiment is reached by the pumping of the ac electromagnetic field and it is plausible that it is not in equilibrium.

A strong dependence of the final molecular production on the initial state (in particular on the value of the initial magnetic field) explains why different experimenters obtain different fractions of molecules in the final state even in the adiabatic regime[14,15,16,18].

In experiments, which achieved a significant molecular production, the initial state was indeed close to the FR, whereas the final state was rather far from the resonance. Thus, in a realistic experimental setup the initial value of ϵ is small $|\epsilon_0| \leq \Gamma \epsilon_F \ll \Delta$ and then ϵ increases linearly with time. In this case one can put $t_0 = 0$, and $\epsilon(t) = \dot{\epsilon}t$. Equation (49) turns into the parabolic cylinder equation. Its standard solution $u(t,0)$ has the asymptotic behavior: $|u(\infty,0)|^2 = \exp(-\pi\Delta^2/2\hbar\dot{\epsilon})$. Employing it together with (52) and (58), we arrive at the following number of molecules in the final state:

$$N_m(+\infty) = F_0^2 N_s^{-1} \left[1 - \exp\left(-\pi\Delta^2/2\hbar\dot{\epsilon}\right)\right] \quad (59)$$

Eq. (44) implies that the maximal possible value of F^2 is $N_s N/2$. It corresponds to the complete transformation of atoms into molecules in the adiabatic regime $\dot{\epsilon} \to 0$. Equation (59) looks exactly like the LZ transition probability multiplied by an effective number of pairs. However, in contrast to phenomenological theories[1,3] and the perturbation theory result[25], the coefficient in front of $1/\dot{\epsilon}$ in the exponent does not depend on the initial atomic density. This theoretical prediction can be checked experimentally. The interaction and many-body effects influence the effective number of pairs (pre-exponent), which is proportional to the square of the initial BCS condensate amplitude $F(t_0)$. Finite temperature destroys a fraction of the initial Cooper pairs and decreases the molecular production.

Finally, we consider an inverse process with no fermions, no BCS condensate and only the molecular condensate in the initial state: $\langle \hat{b} \rangle(-\infty) = \sqrt{N/2}$ and sweeping of the magnetic field in the opposite direction. Then at the end of the sweeping the condensate density is determined by the LZ value: $\langle \hat{b} \rangle(+\infty) = \sqrt{N/2}\exp(-\pi\Delta^2/2\hbar\dot{\epsilon})$, whereas the absolute value of the BCS condensate amplitude $\langle \hat{S}_- \rangle$ can be found from the conservation law (57) (for macroscopic condensate amplitude we can neglect the non-commutativity of S_+ and S_-.):

$$|\langle \hat{S}_- \rangle|^2 = \frac{N_s N}{2} \left[1 - \exp\left(-\frac{\pi\Delta^2}{\hbar\dot{\epsilon}}\right)\right] \quad (60)$$

Notice the factor of 2 difference in the exponents of (59) and (60).

An important problem is the reversibility of the process. Eq. (49) for the amplitude b is time-reversible. It means that, if $b(t)$ is a solution of (49) in a time interval (t_1, t_2), then $b^*(\bar{t})$ is also a solution of the same equation with $\epsilon(t)$ replaced by $\epsilon(\bar{t})$, where $\bar{t}$ denotes the time in the same interval passed in opposite direction. Unfortunately, it is not clear how to reverse the phase of the BEC in the experiment. If we simply start with the same number of molecules in the initial state, which we just before obtained after passing the FR, it does not guarantee that we reproduce in the end the state of the Fermi gas from which the molecules were obtained in the first half-cycle. It is obvious for the non-adiabatic process with not too large LZ parameter $\Delta^2/\dot{\epsilon}$. In this

case to achieve reversibility we need not only to reproduce correct ratio of number of molecules to the BCS condensate density, but also their relative phase which is large and quickly varying value. Even in the adiabatic situation the reversibility is scarcely reachable for two reasons. First, all levels except of the ground state have numerous crossings with other levels (see the previous section) and the adiabatic approximation for them is invalid as it was indicated for more general situation by Polkovnikov and Gritsev[40]. But even when we start with the ground state and sweep the magnetic field adiabatically, the asymptotic value of LZ process is reached only after very long time, since the amplitudes of "parasite" states decay slowly, as $1/t$. The real experiment stops at some finite time, and corrections to the LZ result can be sufficiently large. They are unpredictable and represent a source of irreversibility in the experiment.

VI. CONCLUSION

First we estimate numerical values of different constants entering our theory from available experimental data. The numerical value g can be extracted from the experimental data on the magnetic field dependence of the scattering length a near the FR[41] using the well-known relation: $g = \hbar\sqrt{4\pi(a - a_0)\varepsilon/m}$ (a_0 is the scattering length far from resonance). On the other hand g can be estimated theoretically as $g \sim \epsilon_{hf}\sqrt{a_0^3}$, where ϵ_{hf} is the hyperfine energy[33]. Both these estimates give for ^{40}K $g \sim 10^{-28} erg \times cm^{3/2}$ and from eqs. (16) and (17) $\Delta \sim 3 \times 10^{-4} K$. However, eq. (17) overestimates Δ by assuming that the limiting kinetic energy is Δ instead of it being much smaller. A more reliable estimate can be extracted from a comparison of eq. (59) with experimental data by Regal *et al.*[14]. The fitting gives the value $\Delta \sim 10^{-5} K$ for the broad resonance at $B_0 = 224.21 G$ in ^{40}K. The cited measurements were performed at the finite temperature $T \sim T_F/3$, and therefore the corresponding value of Δ is underestimated in comparison to its zero-temperature value. Thus, a reasonable estimate for Δ is between 10^{-5}K and 10^{-4}K. In the cited experiment the magnetic field sweep amplitude was about 12G. It corresponds to an energy scale of about $10^{-3} K$, larger than Δ.

In conclusion, we considered a cooled Fermi-gas in the magnetic field close the broad Feshbach resonance. We showed that for this strong coupling regime simplifying assumptions stemming from the broad resonance conditions allow one to solve both static and dynamic problems of the BEC-BCS conversion exactly. It was demonstrated that in this situation the single mode approximation is appropriate. The neglect of the fermion kinetic energy, being a rough approximation, gives qualitatively correct physical picture. With these two assumptions we derived the Global Spin Model Hamiltonian (23) and its Hilbert space for the static case. For the dynamic conversion problem we solved the Landau-Zener problem for operators.

For the static problems we have found complete spectrum, the number of molecules and its fluctuations, the amplitude of the BCS condensate and its correlation with the amplitude of the BEC condensate at different values of the detuning parameter ϵ (see equations (42), (43), (44), and (46)).

One of the most surprising results is the big size of the Cooper pair in the momentum space and small its size in configurational space. Though we obtained this result in the framework of the GSM, it will remain in more precise models since it is a consequence of a large extension of the hyperfine interaction in momentum space in comparison with the Fermi momentum. Experimentally this theoretical prediction can be verified by the time-flight spectroscopy combined with the spin correlation analysis. After switching off the trap and the driving magnetic field the Cooper pairs become unstable and decay forming the correlated pairs of particles with opposite velocities and parallel spins. The relaxation time is rather long since fermion pair collisions do not produce energy relaxation. The experimental estimate for the relaxation time is in the range from milliseconds to seconds. Therefore, it seems quite feasible to switch off the trap before the quasimolecules relax and observe the correlations of momenta and spins in runaway particles.

For the dynamic problem we have considered molecule formation and dissociation when the magnetic field is swept across the broad Feshbach resonance. The resulting molecular production from initial fermions is described by LZ-like formulae with a strongly renormalized LZ gap independent of the initial fermion density. However, the molecular production strongly depends on the initial value of magnetic field. Though being reversible in principle, the process is irreversible in practice since the preparation of the inverse process requires fixing of uncontrollable phases and slowly decaying corrections to the LZ asymptotics. Another experimentally verifiable prediction is the independence of the coefficient in front of $1/\dot{\epsilon}$ in the LZ exponents for the molecular (59) and the BCS condensate (60) productions of the initial density of atoms (molecules).

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